LOGIC

1. Incoherence and Philosophy

Logic is about when it is wrong or incoherent to accept a bunch of statements and reject some other statement. For example, it is incoherent to accept the statement "It is raining **and** all mammals are animals" and reject the statement "It is rainnig".[1] This example would be trivial, *if* the point were to say that the example was incoherent. But that isn't the point of the example. Let's assume that we know it's incoherent, there is a lot more to be learned about incoherence from this example. For example, we can discuss *why* it is incoherent to accept the statement "It is raining **and** all mammals are animals" and reject the statement "It is raining". There is an important difference between *knowing that* the sky is blue, and *knowing why* the sky is blue. There is the same difference between *knowing that* the above example is incoherent, and *knowing why* the example is incoherent. Logic is about giving an answer as to *why* the above example is incoherent.

One important feature of the above example that leads towards an explanation is that there doesn't seem to be any special aspect of the statement "It is raining" or the statement "All mammals are animals" that explains why this is incoherent. In fact, the above example is incoherent only because of what "**and**" means. This is said to be a matter of the *logical form* of the statements involved. The notions of *incoherence* and *logical form* are used to explain why some statements *follow logically from* others.

Another central concern of logic is to explain what makes an argument a good one as opposed to a bad one. Here "argument" does not mean "proof". A good *argument* for a statement, S, from a bunch of other statements should convince someone believing the

[1] This paper talks a lot about statements. They are written in blue with the exception of exercises.

other statements that S is true. If they accept all the other statements it would be wrong of them to reject S. In the case of the above example, anyone believing the statement "It is raining and all mammals are animals" should also believe the statement "It is raining", or at least it would be wrong for them to reject it.

2. Arguments

2.1. **Premises and Conclusion.** The goal of logic then is to give an explanation of what makes an argument good or bad. This means that it is important to be precise about what an argument is. For our purposes then, the word "argument" is a *technical term.* A technical term is one that is used in a special sense for the purposes at hand. For example, writers use the word "character" in a special sense to mean "fictional person of the story", whereas in other contexts, like school orientations and company picnics, the use of the word "character" is different. For instance, character building exercises have very little to do with fictional persons in stories. In a similar way, botanists use the word "fruit" as a technical term, for a botanist a tomato is a fruit. Most people use the word "fruit" to exclude tomatoes being one of them.

The first set of technical terms to be cleared up are "acceptance" and "rejection". "Accept" and "reject" here have nothing to do with the way one treats gifts or handshakes. To accept is like accepting a religious belief, or a scientific theory. To accept a statement is to be prepared to offer other statements and observations in its favor, and to be prepared to reject it when it is found to be in conflict with other statements that one accepts. To reject a statement is the converse attitude. It is to be committed to rejecting the statements that would commit one to it, and to be prepared to accept that statement should it be shown that it is incoherent to reject it given what other statements one accepts.

Definition 1. An *argument* is a bunch of statements, one of which counts as the *conclusion*, the others of which count as *premises.*

1 contains two as yet undefined parts: (1) the terms "premises" and "conclusion", and (2) the term "statement". The terms "premise" and "conclusion" are discussed in this section. Statements are discussed in section 2.2. In the above example, the statement "It is raining" counted as the *conclusion* of the argument, and "It is raining **and** all mammals are animals " counted as the premise. In a good argument the premises support, or prove the truth of, the conclusion. If an argument is good, then it is incoherent to accept its premises and deny its conclusion.[2] Because we are considering every possible argument, the conclusion does not have to have anything to do with the premises. For now an argument is *any* list of sentences, some of which are the premises, and *one* of which is the conclusion. Arguments are written as a *numbered list*. The *conclusion* is the last statement in the list, and is separated off by a horizontal line. The argument from the example is written in this style as

Argument 1.

(1) It is raining **and** all animals are mammals

---

(2) It is raining

Arguments can have any number of premises,[3] as in Argument 2.

Argument 2.

(1) **If** it is raining **then** the ground will be wet.

(2) It is raining.

---

(3) The ground will be wet.[4]

---

[2]This isn't the only ingredient to a good argument, but it is the only good ingredient to an argument with good *form*.

[3]Some arguments have no premises, but those are not considered in this text.

[4]As a side note, notice that we can see that this is a good argument because it would be incoherent for someone to accept the *premises* and reject the *conclusion*.

There are, however, no arguments with more than one conclusion. It makes no sense to think of an argument with more than one conclusion. Equally, it makes no sense to think of an argument with fewer than one conclusion. The premises of an argument support the conclusion. There is no argument to be made if the premises are not given as supporting anything, and similarly, there is no way to understand how premises could support more than one conclusion at once.

Exercises. In this section decide whether the following are *arguments*. As a secondary exercise, try to say which of the arguments are good.

(a) 1 All alligators are reptiles.

2 All reptiles are cold-blooded.

---

3 All alligators are cold-blooded.

(b) 1 It is raining.

---

2 It is raining and it is snowing.

(c) 1 Two plus two is five.

---

2 Something plus two is five.

(d) 1 Some cats are felines.

2 Some dogs are felines.

---

3 Some felines are cats.

4 Some felines are dogs.

2.2. **Statements.** The term “statement”, though used extensively, has yet to be defined. The word“statement”, like the word “argument” is a technical term. In most contexts a statement is something someone *makes* as when a statement is made in court, or a politician makes a statement on some issue. The meaning used here is much broader. For our purposes a statement is not an *action* but a *thing*.

Definition 2 (Statement)**.** A statement is anything that is capable of accepted or rejected (ultimately of being true or false).

The following are examples of statements:

- It is raining.
- If it rains then the ground will be wet.
- Everyone who owns a donkey feeds it.

Questions and commands are not capable of being accepted or rejected (true or false). Questions get *answered* and commands get *fulfilled*. To accept a question.

Exercises. Say which of the following are statements:

(a) Nine is a prime number.
(b) Is it raining?
(c) Seven
(d) All dogs are mammals.
(e) What time is it?
(f) Open the window.

Importantly, some statements are made up of other statements. For example, the statement "It is raining **and** all mammals are animals" is a statement. It is made up of the statement "It is raining" and the statement "All mammals are animals". It connects them with the expression "**and**". The word "**and**" is an important expression in English with many uses. The use made of it above is to make a new statement out of two other statements. In general, putting an "**and**" in between two statements will produce a new statement. There are many other such expressions in English. Only are few are required here. The following list suffices for most logical and philosophical purposes:

- ... "**and**" ...
- ... " **or**" ...
- "**It is not the case that**" ...
- "**if**" ... "**then**" ...
- ... "**if and only if**" ...

Here the "... " show a blank where a statement can go. Call these special expressions "logical expressions".[5] The strange logical expression "**it is not the case that**" is used instead of the common word "**not**". This is because in English, in the word "**not**" turns a positive statement to a negative one in only some special circumstances. It can be used to turn "it is raining" into the negative statement "it is not raining". But the right negative statement (the one that is true) of the statement "some cats are dogs" is not the statement "some cats are not dogs" (which is also false), but the statement "no cats are dogs" (which is true). The logical expression "**it is not the case that**" works in both cases to turn a statement into its negative.

Any statement that is made up of other statements by means of logical expressions is called a *complex* statement. A statement that is not complex is called *simple.* An example

[5]The logical expressions "**if and only if**" is often written as "**iff**" for short.

of a simple statement is "all cats are felines", a complex expression is "**if** all cats are felines **then** some cats are felines".

Examples.

- "**It is not the case that** it is raining" is a statement made out of the statement "It is raining" and the logical expression "**it is not the case that**".
- "**If** it is raining **then** the ground will be wet" is a statement made out of the statements "it is raining", "the ground will be wet", and the logical expression "**If**"… "**then**"…
- "**It is not the case that** it is raining **and** the ground is wet" is a statement made out of the statements "**It not the case that** it is raining", "the ground is wet", and the logical expression "**and**". Notice that "**It is not the case that** it raining" is also a complex statement.

Exercises. Say what statements the complex statement is made up of and what logical expression is used.

(a) It is raining or it is snowing

(b) If you go to the store or to the movies, then you will have to drive

(c) It is not the case that Fred is running

(d) It is raining, and I'm glad

2.3. **Complex Statements and Truth.** The *truth* of a complex statement importantly depends on the truth of those smaller statements that make it up. For example, the statement "It is raining **and** all animals are mammals" is true if both of the statements "It is raining" and "All animals are mammals" are true, and it is false otherwise, that is, if one of them is false.[6]

[6] All statements are either true or false, and statements that are not true are false, and statements that are not false are true.

As another example, the statement "It is raining **or** all animals are mammals" is true when either the statement "It is raining" or the statement "All animals are mammals" is true, and false otherwise (if both statements are false).

It is possible to say when a statement made up with the logical expression "**and**" is true by saying when the statements that make it up are true. The same goes for other complex statements. It is possible to say when they are true based on when the sentences they are made up of are true.

Exercise. When is the statement "**It is not the case that** it is raining" true? When is it false? Be sure to give the answer in terms of when the statement that makes it up, "it is raining", is true or false.

The same can be done for all the logical expressions, but it is unnecessary to do so here.

3. Logical Form

Recall the first example. It was said that it is incoherent to accept the statement "It is raining **and** all animals are mammals" and reject the statement "It is raining". In this case, the fact that it is incoherent has nothing to do with the particular statement, "It is raining", if that statement were replaced by *any other statement*, "S", it would be incoherent to accept the statement "S **and** all mammals are animals" and reject the statement "S". In fact, the other simple statement "All animals are mammals" can be replaced by *any other statement*, "R", and it would still be incoherent to accept the statement "S **and** R" and reject the statement, "S". This is a special feature of that argument that deserves attention.

The *logical form* of a statement is what is left when each different simple statement is replaced by a different statement letter: "P", "Q", "R", etc. and no two occurrences of the same simple statement are replaced by different statement letters. As an example, the logical form of "It is raining and all animals are mammals" is "P **and** Q" or "Q **and** R", etc. It doesn't matter which letters are used, just so long as they are used consistently. Importantly, the logical form of that statement is *not* "P **and** P". That would be the logical form of the statement "It is raining and it is raining" (a pretty strange statement to make).

Examples.

	Statement	**Logical Form**
(1)	"It is raining"	"P"
(2)	"All cats are mammals **or** all dogs are mammals"	"M **or** N"
(3)	"**If** everyone loves someone **then** someone loves everyone"	"**If** S **then** R"

Exercises. Give the logical form of these statements.

(a) "It is raining **and** the ground is wet"

(b) "**If** it rains, **then** there will be no sun"

(c) "It is raining **or** all mammals are animals"

Just like statements, arguments have logical form too. The logical form of an argument is based completely on the logical forms of the statements that occur in the argument. When finding the logical form of a statement it is necessary to replace the same simple sentences by the same statement letters, and different simple sentences by different statement letters. The same holds for arguments. So if a simple statement, like "It is raining" appears twice in an argument, and it is replaced by "P" the first, time, it must also be replaced by "P" the second. Once a statement letter, like "P" has been used to replace a simple statement, it can be used to replace *only* that simple statement. This will give the *logical form* of the argument. An *argument form* is the result of replacing all the simple statements in the above way with statement letters. A *concrete argument* is an argument where all the simple statements are statements of English.

Example. The argument from of each concrete argument is to its right.

Argument 1.

(1) It is raining **and** all animals are mammals.

---

(2) It is raining

Logical Form of 1.

(1) P **and** Q

---

(2) P

Argument 3.

(1) **If** It is raining **then** the ground will be wet.

(2) The ground will be wet.

---

(3) It is raining

Logical Form of 3.

(1) **If** P **then** Q

(2) Q

---

(3) P

Importantly, the logical form of Argument 1 is *not* either of

(1) P **and** Q

---

(2) R

(1) P **and** P

---

(2) P

While the same statement letter cannot be used for different simple statements in the when giving the form of one concrete argument, it is fine to use it for different another simple statement when considering a different concrete arguments. This is why it is alright to use the statement letter "Q" for the statement "all animals are mammals" in the logical form of 1, and to use it for the statement "the ground will be wet" in the logical form of 3.

Exercises. Give the logical form of the following concrete arguments. Remember we are not yet evaluating whether the arguments are good.

(a) (1) **It is not the case that** all dogs are mammals.
(2) All dogs are mammals **or** some cats are felines.

---

(3) Some cats are felines.

(b) (1) **If it is not the case that** it is raining, **then** we will go to the beach.
(2) **If** we will go to the beach, **then** we will bring a towel.

---

(3) **If it is not the case that** it is raining, **then** we will bring a towel.

(c) (1) It is raining **and** two plus two is five.

---

(2) Two plus two is five.

(d) (1) **If** all mammals are animals, **then** all mammals are whales.
(2) All animals are whales.

---

(3) All mammals are animals.

Many concrete arguments have the same argument form. For instance the following two concrete arguments have the same argument form as Argument 1:

Argument 4.

(1) The sky is blue **and** some cats are felines.

---

(2) The sky is blue

Argument 5.

(1) Two plus two is twenty-two **and** some cats are not felines.

---

(2) Two plus two is twenty-two

There are lots more concrete arguments with that form.[7] Note though Argument 6 does not have the same argument form as Argument 1.

Argument 6.

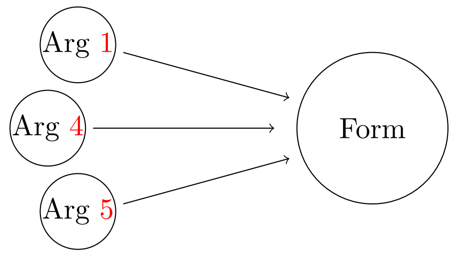
(1) The sky is blue **and** some cats are felines.

---

(2) Some cats are felines.

When considering *argument forms* there are many concrete arguments that have a single form. But when considering a *concrete argument* it has *only one* logical form. The picture looks like this:

[7]Actually there are an infinite number of argument forms and there are an infinite number of concrete arguments of each form.



4. Bad Arguments

There are two ways for a concrete argument to be bad: it could either have a bad logical form, or it could have weak premises. If a concrete argument has a bad form, then every other argument that has that same form will also have a bad form. This is because whether an argument has a bad *form* has nothing at all to do with what its premises and conclusion *actually* are. It has only to do with what the form of the argument is. So in order to check whether or not an argument has bad form, the truth or falsity of its premises or conclusion do not matter at all. The premise and the conclusion of Argument 5 are false, but it has good form. Even though the premise is false, it would be incoherent for someone to accept the premise and reject the conclusion. Suppose someone mistakenly believed the statement "two plus two is twenty-two **and** some cats are not felines". They would be making a *different mistake* if they went on to deny the statement "two plus two is twenty-two" while accepting the statement "two plus two is twenty-two **and** some cats are not felines". It is incoherent to accept the first statement and reject the second.

A concrete argument that has a good logical form, like Argument 5, is said to be *valid*. The definition of validity is given in Definition 3. A concrete argument with a bad form is said to be *invalid*.

4.1. Valid Arguments.

A valid argument is one where it is incoherent to accept the premises and reject the conclusion *based completely on the form of the argument*. The validity of a concrete argument has nothing to do with whether or not the premises of the argument are true or false, or whether the conclusion of the argument is true or false. An argument with false premises can be valid. Consider the silly argument.

Argument 7.

(1) “The earth is flatter than a dime.”

---

(2) “The earth is flatter than a dime.”

7 is valid, even though its premise and its conclusion are false. The reason is that the argument form of Argument 7 is

(1) “P”

---

(2) “P”

It is never coherent to accept any statement “P” and reject that same statement “P”, so that argument must be valid. Its incoherent to accept its premise and reject its conclusion. The reason that it is incoherent to accept the premises of Argument 7 and reject its conclusion is that ideally we reject false things, and ideally we accept true things. If a person were a perfect knower, or were omniscient, they would accept all the true things and deny all the false things. It is *impossible* for any statement “P” to be *both* true and false, and so it would be incoherent to accept and reject the same statement. We can be sure that there is no concrete argument that has the same argument form as Argument 7 with true premises and a false conclusion. No matter what statement of English was used to stand in for “P”, the premise could not be true and the conclusion false.

This can be generalized. An argument form is *good* when no matter what statements of English are used to fill in its statement letters uniformly, it will not be the case that the premises are true and the conclusion is false.

Definition 3 (Valid Argument)**.** A concrete argument, A, is *valid* when there is no concrete argument, B, that has true premises and a false conclusion, and has the same logical form as A.

Definition 4 (Invalid Argument)**.** A concrete argument, A, is *invalid* when there is a concrete argument with true premises and a false conclusion that has the same argument form as A.

When a concrete argument is *invalid* then there is a concrete argument of the same form with true premises and a false conclusion. This concrete argument is a *counter-example* to that argument form. This generates the procedure given by fig. 1 to determine whether or not a concrete argument is valid. A concrete argument is *invalid* when there is a counter-example to its argument form.

As an example take Argument 3.

Argument 3.

(1) **If** It is raining **then** the ground will be wet.

(2) The ground will be wet.

---

(3) It is raining

By **Step 1** of fig. 1, the first thing to do is write down what *argument form* Argument 3 has. The form of Argument 3 is

(1) **If** P **then** Q

(2) Q

---

(3) P

Step 2 of fig. 1 requires us to consider all of the concrete arguments that have that form. Because of space limitations, only one is considered, Argument 8.

Step 1. Find the *form* of the argument.

Step 2. Look at all the other arguments of the same form.

Step 3. If there one of the arguments from **Step 2** has true premises and a false conclusion, then the argument is invalid. Otherwise, the argument is valid.

FIGURE 1. Finding a Counter-example

Argument 8.

(1) **If** George Washington is a cat **then** George Washington is a mammal.

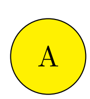
(2) George Washington is a mammal.

---

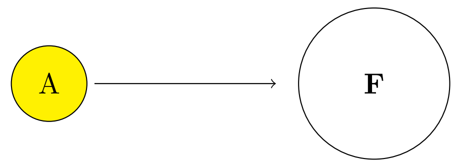
(3) George Washington is a cat.

Step 3 of fig. 1 requires us to consider whether or not there is a *counter-example* to that argument form. Recall a counter-example is a concrete argument of the form in consideration with true premises and a false conclusion. 8 was chosen for the reason that it is a counter-example to that argument form. The first premise of this argument is true because all cats are mammals. So if anything is a cat then that thing is a mammal. Certainly the second premise is true, since the first president of the United States is a human, and all humans are mammals. But the conclusion is false, George Washington is not a cat. 8 is a concrete argument of the *same form* as Argument 3, but it has true premises and a false conclusion. This means that Argument 3 is *invalid*.

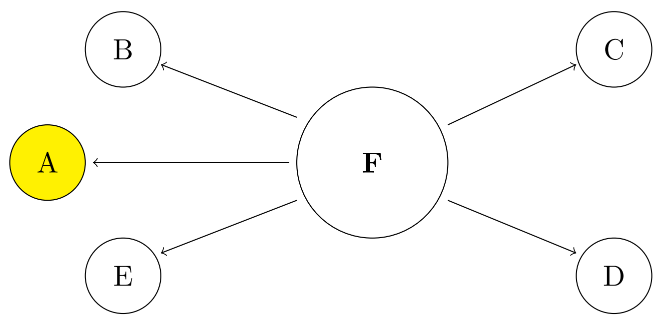
The overall strategy for finding can also be written in this way. Let the yellow circle be the *concrete argument* in question. Call it A



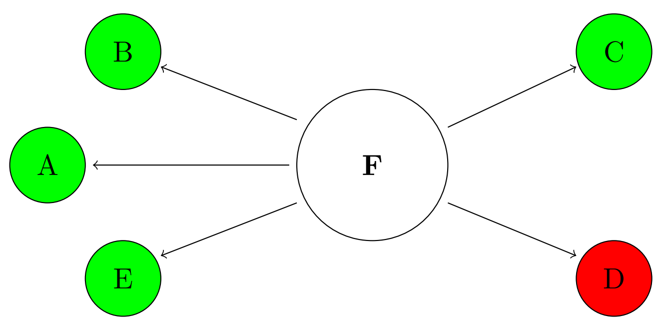
Step 1 requires finding the form of that argument, **F**.



Let the concrete arguments that have form **F** be B, C, D, and E. **Step 2** says to look back at all the concrete arguments that have **F** as their form.



Step 3 requires that A, B, C, D, and E be checked to see if they have true premises and a false conclusion. A circle is filled in red if it has true premises and a false conclusion, and green otherwise.



In the above case D is red. This means that it has true premises and a false conclusion. This means that D is a *counter-example* to the argument form **F**. Since there is a counter-example to A's argument form, A is invalid.

If all the circles above had been green, then there would have been no counter-example to **F**. If there is no counter-example to the argument form of a concrete argument, then that concrete argument is valid. In that case, A would have been *valid*.

Exercises. Find a counter-example to the following arguments. Remember to find the form of the argument first, and then to find a concrete argument of that same form with true premises and a false conclusion.

(a) (1) **It is not the case that** all dogs are cats.

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(2) **It is not the case that** some dogs are cats.

(b) (a) All whales are mammals.

(b) All whales are mammals **and** all cats are felines.

(c) (1) All dogs are mammals **or** two plus two is four.

(2) All dogs are mammals.

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(3) Two plus two is four.

Validity is the most important notion for logicians. When an argument is valid, there is an explanation as to why that argument is good in virtue of its form: no matter what argument has that form, it is impossible for the premises to be true and the conclusion to be false. So if an argument is valid, it is incoherent to accept the premises and reject the conclusion.

4.2. **Factually Incorrect Arguments.** All of the arguments in the previous set of exercises are factually correct. In order for an argument to guarantee that it's conclusion is true, it must be that the information that is used in the argument is accurate. If there is a false premise, then it doesn't matter whether or not the argument is valid. This leads to the following definition:

Definition 5. An argument is *factually correct* when all its premises are true. Otherwise, it's *factually incorrect*.

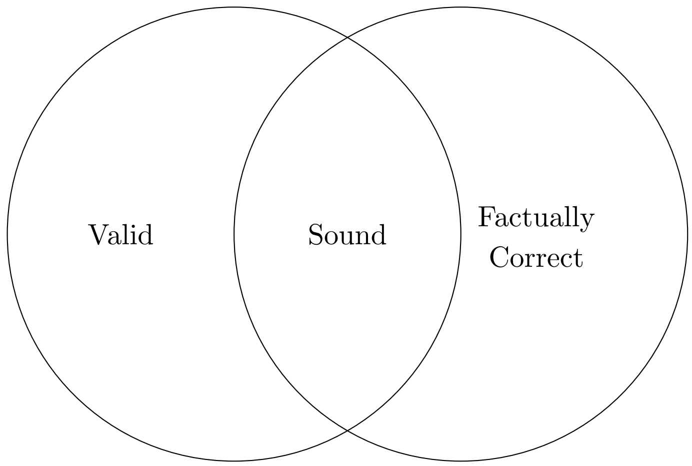
Factual correctness is not of much concern to logicians. What logicians are trying to do is explain *why* an argument is good. Its for others to find out whether the premises of an argument are true or false, and so to learn about whether the conclusion is true. The subject matter of logic is the arguments themselves, and so validity is studied by logicians. Factual correctness is studied by mathematicians, scientists, or philosophers, when the subject matter of the argument is math, science, or philosophy. In fact, in most such areas, with the exception of mathematics, factual correctness is less sought after than good reason to believe. In this case, having a valid argument means that if there is good reason to believe its premises then there is good reason not to doubt the conclusion. In other cases having a valid argument may mean that if it is rational to accept the premises then it is irrational to reject the conclusion. This latter way is of particular interest to philosophers. More often than not, deciding whether the premises are *true* in an important or controversial argument will be very difficult. In such cases, the best that can be done is to settle for rational acceptability or reason to believe.

5. Sound Arguments

If an argument is both valid and factually correct, then its conclusion must be true. This is why the best sort of argument is one that is both valid and factually correct. There is no room to doubt its conclusion. In this case the argument is called *sound*.

Definition 6 (Sound)**.** A concrete argument is *sound* if it is both valid and factually correct.

The following venn diagram sums up the relation between valid, factually correct, and sound arguments



So all sound arguments are both valid and factually correct. There are some arguments, like Argument 8, that are factually correct but not valid, and so not sound. There are other arguments, like Argument 5, that are valid but not factually correct, and so not sound. Finally, there are some arguments that are neither. One example of such an argument is Argument 9.

Argument 9.

(1) The sky is never blue.

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(2) The sky is never blue **and** the sky is blue.

The upshot of all of this is a recipe for what to do when asked to evaluate the conclusion of an argument. The first thing to do is check if the argument is valid. If it is not, then the premises provide no support for the conclusion. After all, it is coherent to accept the premises and deny the conclusion. If the argument is valid, then it must be checked whether or not the premises are true, rational to accept, or reasonable to believe. If not, then even though the argument is valid, there is no problem rejecting the conclusion, so long as one of the premises is also rejected. If, on the other hand, the argument is factually correct, then it is irrational to reject the conclusion. The conclusion is required to be accepted by logic, and the facts on the ground (the truth of the premises).